Consolidated Week 4 Notes: MAST30026 Aug 24, 2012

Lecture 1:

*Def^{n}: Let (X,d) be a metric space. A Cauchy sequence in X is a sequence $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $m,n \in \mathbb{Z}_{>0}$ & $m>N$ & $n>N$ then $d(x_m, x_n) < \varepsilon$

*Def^{n}: A convergent sequence is a sequence $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ such that there exists $z \in X$ such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ & $n>N$ then $d(x_n, z) < \varepsilon$.

* HW: If $x_1, x_2, \ldots$ is a convergent sequence, then $x_1, x_2, \ldots$ is a Cauchy sequence.

* HW: Give an example of a Cauchy sequence that is not a convergent sequence.

– Examples: (1) Let $X = (0,1]$ with $d(x,y) = |y-x|$ & $\vec{x} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

$\vec{x}$ is a Cauchy sequence, but not convergent since $\lim_{n\to\infty} x_n = 0 \notin X$.

(2) $\vec{x} = (3, 3.1, 3.14, 3.141, \ldots)$ is a Cauchy sequence in $\mathbb{Q}$ but is not convergent in $\mathbb{Q}$

*Def^{n}: A metric space is complete if every Cauchy sequence in X is a convergent sequence.

** Theorem: $\mathbb{R}$ with metric $d(x,y) = |y-x|$ is complete

$\mathbb{Q}$ with metric $d(x,y) = |y-x|$ is not complete

Note: A proof of the first statement requires a definition of $\mathbb{R}$.

* HW: Let (X,d) be a metric space & let $Y \subseteq X$. Show that if X is complete & Y is closed then Y is complete.

* HW: Let (X,d) be a metric space & let $Y \subseteq X$. Show that if Y is complete then Y is closed.

* HW: Show that $\mathbb{R}$ is complete. (Only with the standard metric).

* HW: Show that if $X_1, X_2, \ldots, X_m$ are complete metric spaces then $X_1 \times X_2 \times \cdots \times X_m$ is a complete metric space.

– Example: As a metric space $\mathbb{C} = \mathbb{R}^2$ & so $\mathbb{C}$ is complete.

* HW: Let X & Y be metric spaces and let $C_b(X,Y) = \{f: X \to Y \mid f \text{ is continuous} \text{ \& bounded}\}$ with norm $\rho: C_b(X,Y) \times C_b(X,Y) \to \mathbb{R}_{\geq 0}$ given by $\rho(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}$.

Show that if Y complete then $C_b(X, Y)$ is complete.

*Def^n: Let (X, τ) & $(Y, \mathcal{R})$ be topological spaces. A homeomorphism is a function $\varphi: X \to Y$ such that

(a) φ is continuous

(b) $\varphi^{-1}: Y \to X$ exists (φ is a bijection)

(c) $\varphi^{-1}: Y \to X$ is continuous.

*Def^n: Let (X, d) & (Y, ρ) be metric spaces. An isometry from X to Y is a function $f: X \to Y$ such that

if $x_1, x_2 \in X$ then $\rho(f(x_1), f(x_2)) = d(x_1, x_2)$

*Def^n: A function $f: X \to Y$ is injective if it satisfies

if $x_1, x_2 \in X$ & $f(x_1) = f(x_2)$ then $x_1 = x_2$.

*Def^n: A function $f: X \to Y$ is surjective if it satisfies

if $y \in Y$ then there exists $x \in X$ such that $f(x) = y$.

Note: A bijective function is both injective & surjective.

* HW: Show that if $\varphi: X \to Y$ is an isometry then φ is injective.

- Example: An example of an isometry which is not surjective is

$$\varphi: \mathbb{Q} \to \mathbb{R}, \quad x \mapsto x$$

*Def^n: Let (X, d) be a metric space. The completion of (X, d) is a metric space $(\hat{X}, \hat{d})$ with an isometry $\varphi: X \to \hat{X}$ such that $(\hat{X}, \hat{d})$ is complete & $\overline{\varphi(X)} = \hat{X}$

**Theorem: If (X, d) is a metric space then the completion $(\hat{X}, \hat{d})$ exists & is unique.

- Example: $\mathbb{R} = \hat{\mathbb{Q}}$

Lecture 2:

*Def^n: A complete metric space is a metric space (X, d) such that every Cauchy sequence converges.

*Def^n: A complete metric space is a metric space (X, d) that satisfies

if $\vec{x} = (x_1, x_2, \ldots)$ is a sequence in X which satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $m, n \in \mathbb{Z}_{>0}$ & $n > N$ & $m > N$ then $d(x_m, x_n) < \varepsilon$,

then there exists $z \in X$ such that

if $\gamma \in \mathbb{R}_{>0}$ then there exists $M \in \mathbb{Z}_{>0}$ such that

if $k \in \mathbb{Z}_{>0}$ & $k > M$ then $d(x_k, z) < \gamma$.

* HW: $\mathbb{Q}$ is not complete

$\mathbb{R}$ is complete and $\mathbb{Q} \hookrightarrow \mathbb{R}$ & $\overline{\mathbb{Q}} = \mathbb{R}$

Note: $\mathbb{Q} \hookrightarrow \mathbb{C}$ but $\mathbb{C}$ is not a completion of $\mathbb{Q}$ since $\overline{\mathbb{Q}} \neq \mathbb{C}$.

* Def ?: Let (X, d) be a metric space. A completion of (X, d) is a metric space $(\hat{X}, \hat{d})$ with a function $\varphi: X \longrightarrow \hat{X}$ such that

(a) $(\hat{X}, \hat{d})$ is a complete metric space

(b) φ is an isometry

(c) $\overline{\varphi(X)} = \hat{X}$

Note: " $\hat{X}$ is the smallest complete space that contains X "

* HW: Let (X, d) be a metric space. If $(\hat{X}_1, \hat{d}_1)$ with $\varphi_1 : X \longrightarrow \hat{X}_1$ & $(\hat{X}_2, \hat{d}_2)$ with $\varphi_2 : X \longrightarrow \hat{X}_2$ are completions of (X, d), then there exists $f: \hat{X}_1 \longrightarrow \hat{X}_2$ such that

(a) f is an isometry

(b) f is a bijection

(c) $f \circ \varphi_1 = \varphi_2$

** Theorem: $(\hat{X}, \hat{d})$ with $\varphi: X \to \hat{X}$ is a completion of (X, d)

Lecture 3:

- Examples: (1) $\mathbb{R}$ is the completion of $\mathbb{Q}$, $\mathbb{Q} \hookrightarrow \mathbb{R}$, $\overline{\mathbb{Q}} = \mathbb{R}$

(2) $\mathbb{R}[x]$ = polynomials in one variable x

$$\mathbb{R}[x] = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_l x^l \mid l \in \mathbb{Z}_{>0} \ \& \ a_i \in \mathbb{R} \}$$

$$= \{ \textstyle\sum_{i \in \mathbb{Z}_{>0}} a_i x^i \mid a_i \in \mathbb{R} \ \& \text{ all but a finite number of } a_i \text{ are } 0 \}$$

$\mathbb{R}[[x]] = \{ \sum_{i \in \mathbb{Z}_{>0}} a_i x^i \mid a_i \in \mathbb{R} \}$ is the ring of formal power series.

* HW: Show that $\mathbb{R}[[x]]$ is a completion of $\mathbb{R}[x]$, $\mathbb{R}[x] \hookrightarrow \mathbb{R}[[x]]$ and $\overline{\mathbb{R}[x]} = \mathbb{R}[[x]]$.

- Examples: (3) Let $p \in \mathbb{Z}_{>0}$ be a prime.

If $n \in \mathbb{Z}$ then $n = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots$

If $p = 3$ then $56 = 2 \cdot 3^3 + 2 \cdot 3^0 = 2 \cdot 3^0 + 2 \cdot 3^3$

If $p = 2$ then $8 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$

So $8 \equiv 1\,000$

$\mathbb{Z} = \{ a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \{0, 1, \ldots, p-1\}$ & all but finitely many a_i are $0\}$

$= \{ \sum_{i \in \mathbb{Z}_{\geq 0}} a_i p^i \mid a_i \in \{0, 1, \ldots, p-1\}$ & all but finitely many a_i are $0\}$

Then introduce

$\mathbb{Z}_p = \{ \sum_{i \in \mathbb{Z}_{\geq 0}} a_i p^i \mid a_i \in \{0, 1, \ldots, p-1\}\}$ = the p-adic integers

↳ $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$.

Elements of $\mathbb{R}$ are decimal expansions:

$3.1415926\ldots = 3 \cdot 10^0 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + \cdots$

$\in \{ \sum_{i \in \mathbb{Z}} a_i \left(\frac{1}{10}\right)^i \mid a_i \in \{0, 1, \ldots, 9\}\}$

$\mathbb{Q} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}_p$ & $b \neq 0\}$ with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

$\mathbb{Q}_p$ is the p-adic numbers

$\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ and $\overline{\mathbb{Q}} = \mathbb{Q}_p$ in the p-adic metric

End of Week 4.